

CONSISTENT ORIENTATION OF MODULI SPACES

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For Nigel

ABSTRACT. We give an *a priori* construction of the two-dimensional reduction of three-dimensional quantum Chern-Simons theory. This reduction is a two-dimensional topological quantum field theory and so determines to a Frobenius ring, which here is the twisted equivariant K -theory of a compact Lie group. We construct the theory via correspondence diagrams of moduli spaces, which we “linearize” using complex K -theory. A key point in the construction is to *consistently* orient these moduli spaces to define pushforwards; the consistent orientation induces twistings of complex K -theory. The Madsen-Tillmann spectra play a crucial role.

In a series of papers [FHT1, FHT2, FHT3] we develop the relationship between positive energy representations of the loop group of a compact Lie group G and the twisted equivariant K -theory $K_G^{\tau+\dim G}(G)$. Here G acts on itself by conjugation. The loop group representations depend on a choice of “level”, and the twisting τ is derived from the level. For all levels the main theorem is an isomorphism of abelian groups, and for special transgressed levels it is an isomorphism of *rings*: the fusion ring of the loop group and $K_G^{\tau+\dim G}(G)$ as a ring. For G connected with $\pi_1 G$ torsionfree we prove in [FHT1, §4] and [FHT4, §7] that the ring $K_G^{\tau+\dim G}(G)$ is a quotient of the representation ring of G and we calculate it explicitly. In these cases it agrees with the fusion ring of the corresponding centrally extended loop group. We also treat $G = SO_3$ in [FHT4, (A.10)]. In this paper we explicate the multiplication on the twisted equivariant K -theory for an arbitrary compact Lie group G . We work purely in topology; loop groups do not appear. In fact, we construct a *Frobenius ring* structure on $K_G^{\tau+\dim G}(G)$. This is best expressed in the language of topological quantum field theory: we construct a two-dimensional TQFT over the integers in which the abelian group attached to the circle is $K_G^{\tau+\dim G}(G)$.

At first glance the ring structure seems apparent. The multiplication map $\mu: G \times G \rightarrow G$ induces a pushforward on K -theory: the Pontrjagin product. But in K -cohomology the pushforward is the wrong-way, or *umkehr*, map. Thus to define it we must K -orient the map μ . Furthermore, the twistings must be accounted for in the orientations. Finally, to ensure associativity we must consistently K -orient maps constructed from μ by iterated composition. For connected and simply connected groups there is essentially a unique choice, but in general one must work more. This orientation problem is neatly formulated in the language of topological quantum field theory. Cartesian products of G then appear as moduli spaces of flat connections on surfaces, and the maps along which we push forward are restriction maps of the connections to the boundary. What is required, then, is a consistent orientation of these moduli spaces and restriction maps. The existence of consistent orientations, which we prove in Theorem 3.24, is in some sense due to the Narasimhan-Seshadri

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theorem which identifies moduli spaces of flat connections with complex manifolds of stable bundles: complex manifolds carry a canonical orientation in K -theory. Our proof, though, uses only the much more simple linear statement that the symbol of the de Rham complex on a surface is the complexification of the symbol of the Dolbeault complex. As we explain in §1, which serves as a heuristic introduction and motivation, ‘consistent orientations on moduli spaces’ is the topological analog of ‘consistent measures on spaces of fields’ in quantum field theory. The latter is what one would like to construct in the path integral approach to quantum field theory.

Our topological construction, outlined in §3, proceeds via a universal orientation (Definition 3.7). The main observation is that the problem of consistent orientations is a bordism problem, and the relevant bordism groups are those constructed by Madsen and Tillmann [MT] in their formulation of the Mumford conjectures; see [MW, GMTW] for proofs and generalizations. A universal orientation induces a level (Definition 3.23). The map from universal orientations to levels is an isomorphism for simply connected and connected compact Lie groups G , but in general it may fail to be injective, surjective, or both. The theories we construct are parametrized by universal orientations, not by levels. It is interesting to ask whether universal orientations also appear in related topological and conformal field theories as a refinement of the level.

The two-dimensional TQFT we construct here is the dimensional reduction of three-dimensional Chern-Simons theory, refined to have base ring \mathbb{Z} in place of \mathbb{C} . Our construction is *a priori* in the sense that the axioms of TQFT—the topological invariance and gluing laws—are deduced directly from the definition. By contrast, rigorous constructions of many other TQFTs, such as the Chern-Simons theory, proceed via *generators and relations*. Such constructions are based on general theorems which tell that these generators and relations generate a TQFT: gluing laws and topological invariance are satisfied. One can ask if there is an *a priori* topological construction of Chern-Simons theory using twisted K -theory. We do not know of one. In another direction we can extend TQFTs to lower dimension, so look for a theory in 0-1-2 dimensions which extends the 1-2 dimensional theory constructed here. Again, we do not know if there is an *a priori* construction of that extended theory.

Section 2 of this paper is an exposition of twistings and orientation, beginning on familiar ground with densities in differential geometry. Section 4 briefly considers this TQFT for *families* of 1- and 2-manifolds. Our purpose is to highlight an extra twist which occurs: that theory is “anomalous”.

As far as we know, the problem of consistently orienting moduli spaces first arises in work of Donaldson [D, DKr]. He works with anti-self-dual connections on a 4-manifold and uses excision in index theory to relate all of the different moduli spaces. In both his situation and ours the moduli spaces in question sit inside infinite dimensional function spaces, and the virtual tangent bundle to the moduli space extends to a virtual bundle on the function space. Thus it suffices to orient over the function space, and this becomes a universal problem. Presumably our methods apply to his situation as well, but we have not worked out the details.

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It is a pleasure and an honor to dedicate this paper to Nigel Hitchin. We greatly admire his mathematical taste, style, and influence. *¡Feliz cumpleaños y que cumplas muchos más!*

1. PUSH-PULL CONSTRUCTION OF TQFT

Quantum Field Theory

The basic structure of an n -dimensional Euclidean quantum field theory may be axiomatized simply. Let \mathcal{BRiem}_n be the bordism category whose objects are closed oriented $(n-1)$ -dimensional Riemannian manifolds. A morphism $X: Y_0 \rightarrow Y_1$ is a compact oriented n -dimensional Riemannian manifold X together with an orientation-preserving isometry of its boundary to the disjoint union $-Y_0 \sqcup Y_1$, where $-Y_0$ is the oppositely oriented manifold. We term Y_0 the *incoming* boundary and Y_1 the *outgoing* boundary. A quantum field theory is a functor from \mathcal{BRiem}_n to the category of Hilbert spaces and trace class maps. The functoriality encodes the gluing law; there is also a symmetric monoidal structure which encodes the behavior under disjoint unions. There are many details and subtleties (see [S1] in this volume, for example), but our concern is a simpler topological version. Thus we replace \mathcal{BRiem}_n by the bordism category \mathcal{BSO}_n of smooth oriented manifolds and consider orientation-preserving diffeomorphisms in place of isometries. We define an n -dimensional *topological quantum field theory* (TQFT) to be a functor from \mathcal{BSO}_n to the category of complex vector spaces. The functor is required to be monoidal: disjoint unions map to tensor products. The functoriality expresses the usual gluing law and the structure of the domain category \mathcal{BSO}_n encodes the topological invariance. The example of interest here has an integral structure: the codomain is the category of abelian groups rather than complex vector spaces. The integrality reflects that the theory is a dimensional reduction; see [F] for a discussion.

Physicists often employ a path integral to construct a quantum field theory. Here is a cartoon version. To each manifold M is attached a space \mathcal{F}_M of fields and so to a bordism $X: Y_0 \rightarrow Y_1$ a correspondence diagram

$$(1.1) \quad \begin{array}{ccc} & \mathcal{F}_X & \\ s \swarrow & & \searrow t \\ \mathcal{F}_{Y_0} & & \mathcal{F}_{Y_1} \end{array}$$

in which s, t are restriction maps. The important property of fields is locality: in the diagram

$$(1.2) \quad \begin{array}{ccccc} & & \mathcal{F}_{X' \circ X} & & \\ & r \swarrow & & \searrow r' & \\ & \mathcal{F}_X & & \mathcal{F}_{X'} & \\ s \swarrow & & t \searrow & s' \swarrow & \searrow t' \\ \mathcal{F}_{Y_0} & & \mathcal{F}_{Y_1} & & \mathcal{F}_{Y_2} \end{array}$$

the space of fields $\mathcal{F}_{X' \circ X}$ on the composition of bordisms $X: Y_0 \rightarrow Y_1$ and $X': Y_1 \rightarrow Y_2$ is the fiber product of the maps t, s' . Fields are really infinite dimensional *stacks*—for example, in gauge theories the gauge transformations act as morphisms of fields—and the maps and fiber products must be understood in that sense.

The backdrop for the path integral is measure theory. If there exist measures μ_X, μ_Y on the spaces $\mathcal{F}_X, \mathcal{F}_Y$ with appropriate gluing properties, then one can construct a quantum field theory. Namely, define the Hilbert space $\mathcal{H}_Y = L^2(\mathcal{F}_Y, \mu_Y)$ and the linear map attached to a bordism $X: Y_0 \rightarrow Y_1$ as the *push-pull*

$$Z_X = t_* \circ s^*: \mathcal{H}_{Y_0} \longrightarrow \mathcal{H}_{Y_1}.$$

The pushforward t_* is integration. Thus if $f \in L^2(\mathcal{F}_{Y_0}, \mu_{Y_0})$ and $g \in L^2(\mathcal{F}_{Y_1}, \mu_{Y_1})$, then

$$\langle \bar{g}, Z_X(f) \rangle_{\mathcal{H}_{Y_1}} = \int_{\mathcal{F}_X} \overline{g(t(\Phi))} f(s(\Phi)) d\mu_X(\Phi).$$

One usually postulates an *action functional* $S_X: \mathcal{F}_X \rightarrow \mathbb{C}$ and a measure $\tilde{\mu}_X$ such that $\mu_X = e^{-S_X} \tilde{\mu}_X$ and the action satisfies the gluing law

$$S_{X' \circ X}(\Phi) = S_X(r(\Phi)) + S_{X'}(r'(\Phi))$$

in (1.2). These measures have not been constructed in most examples of geometric interest.

Topological Construction

Our idea is to replace the infinite dimensional stack \mathcal{F}_X by a finite dimensional stack $\mathcal{M}_X \subset \mathcal{F}_X$ of solutions to a first order partial differential equation and to shift from measure theory to algebraic topology. Examples of finite dimensional moduli spaces \mathcal{M}_X in supersymmetric field theory include anti-self-dual connections in four dimensions and holomorphic maps in two dimensions. From the physical point of view the differential equations are the BPS equations of supersymmetry; from a mathematical point of view they define the minima of a calculus of variations functional. In this paper we consider pure gauge theories. Fix a compact Lie group G and for any manifold M let \mathcal{F}_M denote the stack of G -connections on M . Define \mathcal{M}_M as the stack of *flat* G -connections on M . If we choose a set $\{m_i\} \subset M$ of “basepoints”, one for each component of M , then \mathcal{M}_M is represented by the product of groupoids $\prod_i [\text{Hom}(\pi_1(M, m_i), G) // G]$. A basic property of flat connections is the gluing law (see (1.2)).

Lemma 1.3. *Suppose $X: Y_0 \rightarrow Y_1$ and $X': Y_1 \rightarrow Y_2$ are bordisms of smooth manifolds. Then $\mathcal{M}_{X' \circ X}$ is the fiber product of*

$$\begin{array}{ccc} \mathcal{M}_X & & \mathcal{M}_{X'} \\ & \searrow t & \swarrow s' \\ & \mathcal{M}_Y & \end{array}$$

Roughly speaking, this says that given flat connections on X, X' and an isomorphism of their restrictions to Y , one can construct a flat connection on $X' \circ X$ and every flat connection on $X' \circ X$ comes this way.

Replace the infinite dimensional correspondence diagram (1.1) with the finite dimensional correspondence diagram of flat connections:

$$(1.4) \quad \begin{array}{ccc} & \mathcal{M}_X & \\ s \swarrow & & \searrow t \\ \mathcal{M}_{Y_0} & & \mathcal{M}_{Y_1} \end{array}$$

Whereas the path integral linearizes (1.1) using measure theory, we propose instead to linearize (1.4) using algebraic topology. Let E be a generalized cohomology theory. To every closed $(n-1)$ -manifold we assign the abelian group

$$A_Y = E^\bullet(\mathcal{M}_Y).$$

To a morphism $X: Y_0 \rightarrow Y_1$ we would like to attach a homomorphism $Z_X: A_{Y_0} \rightarrow A_{Y_1}$ defined as the push-pull

$$(1.5) \quad Z_X := t_* \circ s^*: E^\bullet(\mathcal{M}_{Y_0}) \longrightarrow E^\bullet(\mathcal{M}_{Y_1})$$

in E -cohomology. Whereas the path integral requires *measures* consistent under gluing to define integration t_* , in our topological setting we require *orientations* of t consistent with gluing to define pushforward t_* . The *consistency* of orientations under gluing ensures that (1.5) defines a TQFT which satisfies the gluing law (functoriality).

This, then, is the goal of the paper: we formulate the algebro-topological home for consistent orientations and study a particular example. Namely, specialize to $n = 2$ and require that the 1-manifolds Y and 2-manifolds X be oriented. In other words, the domain category of our TQFT is \mathcal{BSO}_2 . For $Y = S^1$ the moduli stack of flat connections is the global quotient

$$\mathcal{M}_Y \cong G//G$$

of G by its adjoint action; the isomorphism is the holonomy of a flat connection around the circle. Take the cohomology theory E to be complex K -theory. The resulting two-dimensional TQFT on oriented manifolds is the dimensional reduction of three-dimensional Chern-Simons theory for the group G . In this case there is a map from consistent orientations to “levels” on G ; the level is what is usually used to describe Chern-Simons theory. A two-dimensional TQFT on oriented manifolds determines a Frobenius ring and conversely. The Frobenius ring constructed here is the Verlinde ring attached to the loop group of G . The abelian group A_{S^1} is a *twisted* form of $K(G//G) = K_G(G)$ and its relation to positive energy representations of the loop group is developed in [FHT1, FHT2, FHT3]. In this paper we describe a topological construction of the ring structure.

Remarks

- Let X be the “pair of pants” with the two legs incoming and the single waist outgoing. Then restriction to the outgoing boundary is the map $t: (G \times G)//G \rightarrow G//G$ induced by multiplication $\mu: G \times G \rightarrow G$. So $Z_X = t_* \circ s^*$, which defines the ring structure in a two-dimensional TQFT, is pushforward by multiplication on G . Therefore, we do construct the Pontrjagin product on $K_G^{\tau+\dim G}(G)$ —here τ is the twisting and there is a degree shift as well—and have implicitly used an isomorphism of twistings $\mu^*\tau \rightarrow \tau \otimes 1 + 1 \otimes \tau$ which, since the TQFT guarantees an associative product, satisfies a compatibility condition for triple products. This isomorphism and compatibility are embedded in our consistent orientation construction.
- We do not use the theorem [A] which constructs a two-dimensional TQFT from a Frobenius ring. Rather, our *a priori* construction manifestly produces a TQFT which satisfies the gluing law, and we deduce the Frobenius ring as a derived quantity.
- Three-dimensional Chern-Simons theory is defined on a bordism category of manifolds which carry an extra topological structure. For oriented manifolds this extra structure is described as a trivialization of p_1 or signature, or a certain sort of framing. (For spin manifolds it is described as a string structure or, since we are in sufficiently low dimensions, an ordinary framing.) The two-dimensional reduction constructed here factors through the bordism categories of oriented manifolds.
- The topological push-pull construction extends to families of bordisms parametrized by a base manifold S . A choice of consistent orientation determines this extension to a theory for families of manifolds, albeit an “anomalous” theory; see §4 for a discussion.
- The pushforward t_* is only defined if $t: \mathcal{M}_X \rightarrow \mathcal{M}_{Y_1}$ is a *representable* map of stacks, i.e., only if the fibers of t are spaces—no automorphisms allowed. This happens only if each component of X has a nonempty outgoing boundary. Therefore, the push-pull construction only gives a partial TQFT. We complete to a full TQFT using the nondegeneracy of a certain bi-additive form; see [FHT3, §17].
- As mentioned earlier, a standard TQFT is defined over the ring \mathbb{C} whereas this theory, being a dimensional reduction of a 3-dimensional theory, is defined over \mathbb{Z} . It is possible to go further and refine the push-pull construction to obtain a theory over K , where K is the K -theory ring spectrum. See [F] for further discussion.
- The theory constructed here has two tiers—it concerns 1- and 2-manifolds—so could be termed a ‘1-2 theory’. Extensions to 0-1-2 theories, which have three tiers, are of great interest. The general structure of such theories has been much studied recently in various guises [MS], [C], [HL]. A theory defined down to points is completely local, and so ultimately has a simpler structure than less local theories. We do not know if the push-pull construction here can be extended to construct a 0-1-2 theory.

2. ORIENTATION AND TWISTING

Ordinary Cohomology

The first example for a differential geometer is de Rham theory. Let M be a smooth manifold and suppose it has a dimension equal to n . An *orientation* on M , which is an orientation of the tangent bundle TM , enables integration

$$\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$$

on forms of compact support. Absent an orientation we can integrate twisted forms, or densities. The *twisting* is defined as follows. For any real vector space V of dimension r let $\mathcal{B}(V)$ denote the $GL_r\mathbb{R}$ -torsor of bases of V . There is an associated \mathbb{Z} -graded real line $\mathfrak{o}(V)$ of functions $f : \mathcal{B}(V) \rightarrow \mathbb{R}$ which satisfy $f(b \cdot A) = \text{sign det } A \cdot f(b)$ for $b \in \mathcal{B}(V)$, $A \in GL_r\mathbb{R}$; the degree of $\mathfrak{o}(V)$ is r . Applied fiberwise this construction yields a flat \mathbb{Z} -graded line bundle $\mathfrak{o}(V) \rightarrow M$ for a real vector bundle $V \rightarrow M$. There is a twisted de Rham complex

$$(2.1) \quad 0 \longrightarrow \Omega^{\mathfrak{o}(V)-r}(M) \xrightarrow{d} \Omega^{\mathfrak{o}(V)-r+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{\mathfrak{o}(V)}(M) \longrightarrow 0$$

where $\Omega^{\mathfrak{o}(V)+q}(M)$ is the space of smooth sections of the *ungraded* vector bundle $\bigwedge^{r+q} T^*M \otimes \mathfrak{o}(V)$. The cohomology of (2.1) is the twisted de Rham cohomology $H_{dR}^{\mathfrak{o}(V)+\bullet}(M)$. Let $\mathfrak{o}(M) = \mathfrak{o}(TM)$. Then integration is a map

$$(2.2) \quad \int_M : \Omega_c^{\mathfrak{o}(M)}(M) \longrightarrow \mathbb{R}.$$

Notice this formulation-notation works if M has several components of varying dimension: the degree of $\mathfrak{o}(M)$ is then the locally constant function $\dim M : M \rightarrow \mathbb{Z}$.

A similar construction works in integer cohomology. If $\pi : V \rightarrow M$ is a real vector bundle over a space M (which needn't be a manifold) we define $\mathfrak{o}(V)$ as the orientation double cover of M determined by V and introduce a \mathbb{Z} -grading according to the rank of V . (Note $\text{rank } V : M \rightarrow \mathbb{Z}$ is a locally constant function.) There is an $\mathfrak{o}(V)$ -twisted singular complex analogous to (2.1): cochains in this complex are cochains on the double cover which change sign under the deck transformation. The equivalence class of the twisting $\mathfrak{o}(V)$ is

$$[\mathfrak{o}(V)] = (\text{rank } V, w_1(V)) \in H^0(M; \mathbb{Z}) \times H^1(M; \mathbb{Z}/2\mathbb{Z}),$$

where w_1 is the Stiefel-Whitney class. The relationship of the twisting to integration occurs in the Thom isomorphism. The Thom class $U \in H_{cv}^{\pi^*\mathfrak{o}(V)}(V)$ lies in twisted cohomology with *compact vertical* support. Let $B(V), S(V)$ be the ball and sphere bundles relative to a metric on V . The *Thom space* M^V is the pair $(B(V), S(V))$ or equivalently, assuming M is a CW complex, the quotient $B(V)/S(V)$. The composite

$$H^{-\mathfrak{o}(V)+\bullet}(M) \xrightarrow{\pi^*} H^{-\pi^*\mathfrak{o}(V)+\bullet}(V) \xrightarrow{U} H^\bullet(M^V)$$

is an isomorphism—the Thom isomorphism—a generalization of the suspension isomorphism. If M is a *compact* manifold and $i: M \hookrightarrow S^n$ a (Whitney) embedding with normal bundle $\nu \rightarrow M$, then the *Pontrjagin-Thom collapse* is the map $c: S^n \rightarrow M^\nu$ defined by identifying ν with a tubular neighborhood of M and sending the complement of $B(\nu)$ in S^n to the basepoint of M^ν . Integration is then the composite

$$(2.3) \quad H^{-\mathfrak{o}(\nu)+n}(M) \xrightarrow[\cong]{\text{Thom}} H^n(M^\nu) \xrightarrow{c^*} H^n(S^n) \xrightarrow[\cong]{\text{suspension}} \mathbb{Z}.$$

Twistings obey a Whitney sum formula: there is a natural isomorphism

$$\mathfrak{o}(V_1 \oplus V_2) \xrightarrow{\cong} \mathfrak{o}(V_1) + \mathfrak{o}(V_2).$$

Applied to $TM \oplus \nu = \underline{n}$, where \underline{n} is the trivial bundle of rank n , we conclude that integration (2.3) is a map (compare (2.2))

$$H^{\mathfrak{o}(M)}(M) \longrightarrow \mathbb{Z}.$$

More generally, if $p: M \rightarrow N$ is a proper map there is a pushforward

$$(2.4) \quad p_*: H^{\mathfrak{o}(p)+\bullet}(M) \longrightarrow H^\bullet(N),$$

where $\mathfrak{o}(p) = \mathfrak{o}(M) - p^*\mathfrak{o}(N)$.

***K*-theory**

This discussion applies to any multiplicative cohomology theory.¹ The only issue is to determine the twisting of a real vector bundle in that theory. For complex *K*-theory there are many possible models for the twisting τ_V of a vector bundle $V \rightarrow M$. In the Donovan-Karoubi [DK] picture τ_V is represented by the bundle of complex $\mathbb{Z}/2\mathbb{Z}$ -graded Clifford algebras defined by V . A bundle of algebras $A \rightarrow M$ of this type is considered trivial if $A = \text{End}(W)$ for a $\mathbb{Z}/2\mathbb{Z}$ -graded complex vector bundle $W \rightarrow M$, i.e., if A is Morita equivalent to the trivial bundle of algebras $M \times \mathbb{C}$. The equivalence class of τ_V is

$$(2.5) \quad [\tau_V] = (\text{rank } V, w_1(V), W_3(V)) \in H^0(M; \mathbb{Z}/2\mathbb{Z}) \times H^1(M; \mathbb{Z}/2\mathbb{Z}) \times H^3(M; \mathbb{Z}).$$

Only torsion classes in $H^3(M; \mathbb{Z})$ are realized by bundles of finite dimensional algebras, but we have in mind a larger model which includes nontorsion classes. (Such models are developed in [AS], [FHT1], [M] among other works.) There is a Whitney sum isomorphism

$$(2.6) \quad \tau_{V_1 \oplus V_2} \xrightarrow{\cong} \tau_{V_1} + \tau_{V_2};$$

¹also to a cohomology theory defined by a module over a ring spectrum.

the sum of twistings is realized by the tensor product of algebras. A spin^c structure on V induces an *orientation*, i.e., a Morita equivalence

$$(2.7) \quad \tau_{\underline{\text{rank}} V} \xrightarrow{\cong} \tau_V.$$

An *A-twisted* vector bundle is a vector bundle with an A -module structure; it represents an element of twisted K -theory.

The Whitney formula (2.6) allows us to attach a twisting to any virtual real vector bundle: set

$$(2.8) \quad \tau_{-V} = -\tau_V.$$

Since the Thom space satisfies the stability condition $X^{V \oplus \mathbb{Z}} \cong \Sigma^n X^V$, where ‘ Σ ’ denotes suspension, there is also a Thom spectrum attached to any virtual vector bundle and a corresponding Thom isomorphism theorem. An orientation, which is an isomorphism as in (2.7), is equivalently a trivialization of the twisting attached to the reduced bundle $(V - \underline{\text{rank}} V)$.

Remark 2.9. There are also twistings of K -theory—indeed of any cohomology theory—which do not come from vector bundles.

Suppose τ is any twisting on a manifold N . We can put that extra twisting into the pushforward in K -theory associated to a proper map $p: M \rightarrow N$ (compare (2.4)):

$$(2.10) \quad p_*: K^{(\tau_p + p^*\tau) + \bullet}(M) \longrightarrow K^{\tau + \bullet}(N).$$

Here τ_p is the twisting $\tau_p = \tau_M - p^*\tau_N$ of the relative tangent bundle. In the next section we encounter a situation in which $\tau_p + p^*\tau$ is trivialized, and so construct a pushforward from untwisted K -theory to twisted K -theory; see (3.31).

Twistings of $K^\bullet(\text{pt})$ form a symmetric monoidal 2-groupoid; its classifying space $\text{Pic}_g K$ is thus an infinite loop space. The notation: $\text{Pic} K$ is the classifying space of invertible K -modules and $\text{Pic}_g K$ the subspace classifying certain “geometric” invertible K -modules including twisted forms of K -theory defined by real vector bundles. As a space there is a homotopy equivalence

$$(2.11) \quad \text{Pic}_g K \sim K(\mathbb{Z}/2\mathbb{Z}, 0) \times K(\mathbb{Z}/2\mathbb{Z}, 1) \times K(\mathbb{Z}, 3)$$

with a product of Eilenberg-MacLane spaces, but the group structure on $\text{Pic}_g K$ is not a product. The group of equivalence classes of twistings on M is the group of homotopy classes of maps $[M, \text{Pic}_g K]$, which as a set is the product of cohomology groups in (2.5).

Let $\text{pic}_g K$ denote the spectrum whose 0-space is $\text{Pic}_g K$ and which is a Postnikov section of the real KO -theory spectrum: the “ $\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}$ ” bit of the “ $\dots, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots$ ” song. Thus the 1-space $B \text{Pic}_g K$ of the spectrum $\text{pic}_g K$ is a Postnikov section of BO . Also, let ko denote the connective KO -theory spectrum. Its 0-space is the group completion of the classifying space of the symmetric monoidal category of real vector spaces of finite dimension [S2]. The map which attaches a twisting of $K^\bullet(\text{pt})$ to a real vector space, say via the Clifford algebra, induces a spectrum map

$$(2.12) \quad \tau: ko \longrightarrow \text{pic}_g K.$$

Remark 2.13. If M is a smooth *stack*, then its tangent space is presented as a graded vector bundle. An orientation of M is then an orientation of this graded bundle, viewed as a virtual bundle by taking the alternating sum of the homogeneous terms. The virtual tangent bundle is classified by a map $M \rightarrow ko$ whose composition with (2.12) gives the induced twisting of complex K -theory. We apply this in §3 to the moduli space of flat connections on a fixed oriented 2-manifold. In that case the virtual tangent bundle is the index of an elliptic complex and the map $M \rightarrow ko$ is computed from the Atiyah-Singer index theorem [ASi].

The \mathbb{Z} part of $\text{Pic}_g K$, the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$, has an attractive geometric model: gerbes. Nigel Hitchin has developed beautiful applications of gerbes in differential geometry [H]. There is a gerbe model for $\text{Pic}_g K$ too—one need only add $\mathbb{Z}/2\mathbb{Z}$ -gradings.

3. UNIVERSAL ORIENTATIONS AND CONSISTENT ORIENTATIONS

Overview

In this section, the heart of the paper, we define universal orientations (Definition 3.7) and prove that they exist (Theorem 3.24). A universal orientation simultaneously orients the maps t in (1.4) along which we pushforward classes in twisted K -theory; see (3.33) for the precise push-pull maps in the theory. Universal orientations form a torsor for an abelian group (3.9). A universal orientation determines a level (Definition 3.23), which is the quantity typically used to label theories. The map (3.27) from universal orientations to levels is not an isomorphism in general.

We begin with a closed oriented surface X . The virtual tangent space to the stack \mathcal{M}_X of flat G -connections on X is the index of a twisted de Rham complex (3.1), and we construct a universal symbol (3.4)—whence universal index—for this operator. A trivialization of the universal twisting (3.8) is a universal orientation, and it simultaneously orients the moduli stacks \mathcal{M}_X for all closed oriented X .

For a surface X with (outgoing) boundary we must orient the restriction map $t: \mathcal{M}_X \rightarrow \mathcal{M}_{\partial X}$ on flat connections. It turns out that a universal orientation does this, simultaneously and coherently for all X , as expressed in the isomorphism (3.30). An important step in the argument is Lemma 3.19, which uses work of Atiyah-Bott [AB] to interpret the universal symbol in terms of standard local boundary conditions for the de Rham complex.

Closed surfaces

Fix a compact Lie group G with Lie algebra \mathfrak{g} . Let X be a *closed* oriented 2-manifold and \mathcal{M}_X the moduli stack of flat G -connections on X . A point of \mathcal{M}_X is represented by a flat connection A on a principal bundle $P \rightarrow X$, and the tangent space to \mathcal{M}_X at A by the deformation complex

$$(3.1) \quad 0 \longrightarrow \Omega_X^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega_X^1(\mathfrak{g}_P) \xrightarrow{d_A} \Omega_X^2(\mathfrak{g}_P) \longrightarrow 0,$$

the de Rham complex with coefficients in the adjoint bundle associated to P . This is an elliptic complex. Its symbol σ satisfies the reality condition $\sigma(-\xi) = \overline{\sigma(\xi)}$ for $\xi \in TX$, since (3.1) is

a complex of real differential operators [ASi]. Recall that the symbol of any complex differential operator lies in $K_{\text{cv}}^0(TX) \cong K^0(X^{TX})$. The reality condition gives a lift $\sigma \in KR^0(X^{iTX})$, where the imaginary tangent bundle iTX carries the involution of complex conjugation [At]. Bott periodicity asserts that $V \oplus iV$ is canonically KR -oriented for any real vector bundle V with no degree shift. In the language of twistings of KR this means

$$(3.2) \quad \tau_V^{(KR)} + \tau_{iV}^{(KR)} = 0.$$

Therefore

$$(3.3) \quad KR^0(X^{iTX}) \xrightarrow[\cong]{\text{Thom}} KR^{-\tau_{iTX}^{(KR)}}(X) \xrightarrow[\cong]{(3.2)} KR^{\tau_{TX}^{(KR)}}(X) \xrightarrow[\cong]{(2.8)} KR^{-\tau_{-TX}^{(KR)}}(X) \xrightarrow[\cong]{\text{Thom}} KO^0(X^{-TX})$$

from which we locate the symbol $\sigma \in KO^0(X^{-TX})$. Note that by Atiyah duality $KO^0(X^{-TX}) \cong KO_0(X)$ and the KO -homology group is well-known to be the home of real elliptic operators.

Now (3.1) is a *universal* operator: its symbol is constructed from the exterior algebra of TX and the adjoint representation of G ; it does not depend on details of the manifold X . Thus it is pulled back from a universal symbol. Let $V_n \rightarrow BSO_n$ denote the universal oriented n -plane bundle. The universal symbol lives on the bundle $V_2 \rightarrow BSO_2 \times BG$, and by (3.3) we identify it as an element

$$(3.4) \quad \sigma_{\text{univ}} \in KO^0(BSO_2^{-V_2} \wedge BG_+).$$

Here BG_+ is the space BG with a disjoint basepoint adjoined and ‘ \wedge ’ is the smash product. Introduce the notation

$$MTSO_n = BSO_n^{-V_n}$$

for this Thom spectrum and so write

$$\sigma_{\text{univ}} \in KO^0(MTSO_2 \wedge BG_+).$$

If $f: X \rightarrow BSO_2 \times BG$ is a classifying map for TX and P , and $\tilde{f}: X^{-TX} \rightarrow MTSO_2 \times BG \rightarrow MTSO_2 \wedge BG$ the induced map on Thom spectra, then $\sigma = \tilde{f}^* \sigma_{\text{univ}}$. It is in this sense that (3.4) is a *universal* symbol.

Remark 3.5. We digress to explain $MTSO_n$ in more detail. Let $Gr_n^+(\mathbb{R}^N)$ be the Grassmannian of oriented n -planes in \mathbb{R}^N and

$$0 \rightarrow V_n \rightarrow \underline{N} \rightarrow Q_{N-n} \rightarrow 0$$

the exact sequence of real vector bundles over $Gr_n^+(\mathbb{R}^N)$ in which V_n is the universal subbundle and Q_{N-n} the universal quotient bundle. Denote the Thom space of the latter as

$$Z_N := Gr_n^+(\mathbb{R}^N)^{Q_{N-n}}.$$

Then the suspension ΣZ_N is the Thom space of $Q_{N-n} \oplus \underline{1} \rightarrow Gr_n^+(\mathbb{R}^N)$. But $Q_{N-n} \oplus \underline{1}$ is the pullback of $Q_{N+1-n} \rightarrow Gr_n^+(\mathbb{R}^{N+1})$ under the natural inclusion $Gr_n^+(\mathbb{R}^N) \hookrightarrow Gr_n^+(\mathbb{R}^{N+1})$, and in this manner

we produce a map $\Sigma Z_N \rightarrow Z_{N+1}$. Whence the spectrum $MTSO_n = \{Z_N\}_{N \geq 0}$. The notation identifies $MTSO_n$ as an unstable version of the Thom spectrum MSO and also alludes to its appearance in the work of Madsen-Tillmann [MT]. There are analogous spectra MTO_n , $MT\text{Spin}_n$, $MT\text{String}_n$, etc. If $F: S^N \rightarrow Z_N$ is transverse to the 0-section, then $X := F^{-1}(0\text{-section})$ is an n -manifold and the pullback of $Q_{N-n} - \underline{N}$ is stably isomorphic to $-TX$. Thus a map² $S \rightarrow MTSO_n$ classifies a map $M \rightarrow S$ of relative dimension n together with a rank n bundle $W \rightarrow M$ and a stable isomorphism³ $T(M/S) \cong W$. An important theorem of Madsen-Weiss [MW] and Galatius-Madsen-Tillmann-Weiss [GMTW] identifies $MTSO_n$ as a bordism theory of fiber bundles rather than a bordism theory of arbitrary maps.

If a *smooth manifold* M parametrizes a family of flat G -connections on X —that is, $P \rightarrow M \times X$ is a G -bundle with a partial flat connection along X —then there is a classifying map $M \rightarrow \mathcal{M}_X$ and the pullback of the stable tangent bundle of \mathcal{M}_X to M is the index of the family of elliptic complexes (3.1). Note that if we replace the adjoint bundle \mathfrak{g}_P in (3.1) by the trivial bundle of rank $\dim G$ then the resulting elliptic complex does not vary over M ; its index is a trivializable bundle. Hence the reduced stable tangent bundle to \mathcal{M}_X is computed by the de Rham complex coupled to the reduced adjoint bundle $\bar{\mathfrak{g}}_P = \mathfrak{g}_P - \underline{\dim G}$.

There is a corresponding reduced universal symbol (compare (3.4))

$$(3.6) \quad \bar{\sigma}_{\text{univ}} \in KO^0(MTSO_2 \wedge BG).$$

It induces a universal twisting in K -theory and a consistent orientation is constructed by trivializing this twisting.

Definition 3.7. *A universal orientation is a null homotopy of the composition*

$$(3.8) \quad MTSO_2 \wedge BG \xrightarrow{\bar{\sigma}_{\text{univ}}} ko \xrightarrow{\tau} \text{pic}_g K.$$

Two universal orientations are said to be equivalent if the null homotopies are homotopy equivalent.

The set of equivalence classes of universal orientations is a torsor for the abelian group

$$(3.9) \quad \mathcal{O}(G) := [\Sigma MTSO_2 \wedge BG, \text{pic}_g K].$$

We prove in Theorem 3.24 below that universal orientations exist. In fact, there is a canonical universal orientation, so the torsor of universal orientations may be naturally identified with the abelian group (3.9). Definition 3.7 is designed to orient the moduli spaces attached to closed surfaces. In an equivalent form it leads to the pushforward maps we need for surfaces with boundary and to twisted K -theory of moduli spaces attached to the boundary; see the discussion preceding (3.20).

Return now to the family of partial G -connections on $P \rightarrow M \times X$. The bundle $P \rightarrow M \times X$ is classified by a map $f: M \rightarrow MTSO_2 \times BG$ and the Atiyah-Singer index theorem [ASi] implies that the index of the family of operators (3.1) is $f^* \sigma_{\text{univ}}$. Therefore, a universal orientation pulls back to an orientation of the index of (3.1); c.f. Remark 2.13. It follows that a universal orientation simultaneously orients \mathcal{M}_X for all closed oriented 2-manifolds X .

²i.e., a stable map from the suspension spectrum of S to $MTSO_n$

³Thom bordism theories, such as MSO , retain the information of the stable normal bundle. Madsen-Tillmann theories, such as $MTSO_n$, track the stable tangent bundle, which is one more justification for the ‘T’ in the notation.

Surfaces with boundary

As a preliminary we observe two topological facts about $MTSO_n$.

Lemma 3.10. (i) $MTSO_1 \simeq S^{-1}$, the desuspension of the sphere spectrum.
 (ii) There is a cofibration

$$(3.11) \quad \Sigma^{-1}MTSO_{n-1} \xrightarrow{b} MTSO_n \twoheadrightarrow (BSO_n)_+.$$

Proof. For (i) simply observe

$$(3.12) \quad MTSO_1 \simeq BSO_1^{-V_1} \simeq \text{pt}^{-\mathbb{R}} \simeq S^{-1}.$$

For (ii) begin with the cofibration built from the sphere and ball bundles of the universal bundle:

$$(3.13) \quad S(V_n)_+ \hookrightarrow B(V_n)_+ \twoheadrightarrow (B(V_n), S(V_n)) ,$$

Then identify BSO_{n-1} as the unit sphere bundle $S(V_n)$ and write (3.13) in terms of Thom spaces:

$$(3.14) \quad BSO_{n-1}^{\underline{0}} \hookrightarrow BSO_n^{\underline{0}} \twoheadrightarrow BSO_n^{V_n} .$$

Here $\underline{0}$ is the vector bundle of rank zero. Now add $-V_n$ to each of the vector bundles in (3.14) and note that the restriction of V_n to BSO_{n-1} is $V_{n-1} \oplus \underline{1}$.

Consider the diagram

$$(3.15) \quad \begin{array}{ccc} \Sigma^{-1}MTSO_1 \wedge BG & \xrightarrow{b} & MTSO_2 \wedge BG \xrightarrow{q} (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG \\ & & \searrow \bar{\sigma}_{\text{univ}} \quad \downarrow \bar{\sigma}'_{\text{univ}} \\ & & ko \end{array}$$

The top row is a cofibration. From (3.11) we can replace $(MTSO_2, \Sigma^{-1}MTSO_1)$ with $(BSO_2)_+$.

Lemma 3.16. Define $\bar{\sigma}'_{\text{univ}}$ in (3.15) to be the map $(BSO_2)_+ \wedge BG \rightarrow ko$ induced by the trivial representation of SO_2 smash with the reduced adjoint representation of G . Then the triangle in (3.15) commutes and the diagram gives a canonical null homotopy of the composite $\bar{\sigma}_{\text{univ}} \circ b$.

Proof. Recalling the isomorphisms in (3.3), and using the fact that the universal symbol $\bar{\sigma}_{\text{univ}}$ is canonically associated to a representation of $SO_2 \times G$, we locate $\bar{\sigma}_{\text{univ}} \in KR_{SO_2 \times G}^0(-\mathbb{R}^2)_c \cong KR_{SO_2 \times G}^0(i\mathbb{R}^2)_c$. (Recall that the involution on $\mathbb{C}^2 \cong \mathbb{R}^2 \oplus i\mathbb{R}^2$ is complex conjugation and the subscript ‘c’ denotes compact support.) Similarly, $\bar{\sigma}'_{\text{univ}} \in KR_{SO_2 \times G}^0(\text{pt}) \cong KR_{SO_2}^0(\mathbb{C}^2)_c$. Using the Thom isomorphism we identify $\bar{\sigma}'_{\text{univ}}$ as the difference of classes represented by $\bigwedge^\bullet \mathbb{C}^2 \otimes \mathfrak{g}_{\mathbb{C}}$ and $\bigwedge^\bullet \mathbb{C}^2 \otimes \mathbb{C}^{\dim G}$, where in both summands $\theta \in \mathbb{C}^2$ acts as $\epsilon(\theta) \otimes \text{id}$. Exterior multiplication $\epsilon(\theta)$ is exact for $\theta \neq 0$, so this does represent a class with compact support. Also, as ϵ commutes with complex conjugation it is Real in the sense of [At]. It remains to observe that its restriction under $KR_{SO_2 \times G}^0(\mathbb{C}^2)_c \rightarrow KR_{SO_2 \times G}^0(i\mathbb{R}^2)_c$ is the universal symbol $\bar{\sigma}_{\text{univ}}$ of the de Rham complex coupled to the reduced adjoint bundle.

For any n a point of the 0-space of the pair of spectra $(MTSO_n, \Sigma^{-1}MTSO_{n-1})$ is represented by a map from (B^N, S^{N-1}) into the pair of Thom spaces attached to

$$\begin{array}{ccc} Q_{N-n} & \longrightarrow & Q_{N-n} \\ \downarrow & & \downarrow \\ Gr_{n-1}^+(\mathbb{R}^{N-1}) & \hookrightarrow & Gr_n^+(\mathbb{R}^N) \end{array}$$

for N sufficiently large. A map of (B^N, S^{N-1}) into this pair which is transverse to the 0-section gives a compact oriented n -manifold M with boundary embedded in (B^N, S^{N-1}) , a rank n bundle $W \rightarrow M$ equipped with a stable isomorphism $TM \cong W$, and a splitting of $W|_{\partial M}$ as the direct sum of a rank $(n-1)$ bundle and a trivial line bundle. The composition with the boundary map

$$(3.17) \quad r: (MTSO_n, \Sigma^{-1}MTSO_{n-1}) \rightarrow MTSO_{n-1}$$

is the restriction of this data to ∂M .

Now let M be a smooth manifold, X a compact oriented 2-manifold with boundary, and $P \rightarrow M \times X$ a principal G -bundle with partial flat connection along X . This data induces a classifying map $M \rightarrow \mathcal{M}_X$ and, forgetting the connection, a classifying map

$$(3.18) \quad f: M \longrightarrow (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG.$$

There are induced classifying maps $M \rightarrow \mathcal{M}_{\partial X}$ and

$$\dot{f}: M \longrightarrow MTSO_1 \wedge BG$$

for the boundary data; here $\dot{f} = r \circ f$. View X as a bordism $X: \emptyset \rightarrow \partial X$; later we incorporate incoming boundary components. The following key result relates the universal topology above to surfaces with boundary.

Lemma 3.19. *The map $\bar{\sigma}'_{\text{univ}} \circ f: M \rightarrow ko$ classifies the reduced tangent bundle of the restriction map $t: \mathcal{M}_X \rightarrow \mathcal{M}_{\partial X}$ —the bundle $T\mathcal{M}_X - t^*T\mathcal{M}_{\partial X}$ reduced to rank zero—pulled back to M .*

Proof. At a point $A \in M$ there is a flat connection on $P|_{\{A\} \times X}$. The tangent space to \mathcal{M}_X at that point is computed by the twisted de Rham complex (3.1), so is represented by the twisted de Rham cohomology $H_A^\bullet(X)$. Similarly, the tangent space to $\mathcal{M}_{\partial X}$ at the restriction $t(A)$ of A to the boundary is $H_{t(A)}^\bullet(\partial X)$. From the long exact sequence of the pair $(X, \partial X)$ we deduce that the difference $T\mathcal{M}_X - t^*T\mathcal{M}_{\partial X}$ at A is the twisted relative de Rham cohomology $H_A^\bullet(X, \partial X)$.

Now the twisted relative de Rham cohomology is the index of the deformation complex (3.1) with *relative boundary conditions* [G, §4.1]. In other words, we consider the subcomplex of forms which vanish when restricted to ∂X . This is an example of a *local* elliptic boundary value problem.

Atiyah and Bott [AB] interpret local boundary conditions in K -theory and prove an index formula. More precisely, the triple $(B(TX), \partial B(TX), S(TX))$ leads to the exact sequence

$$\begin{aligned} KR^{-1}(B(TX)|_{\partial X}, S(TX)|_{\partial X}) &\longrightarrow KR^0(B(TX), \partial B(TX)) \\ &\longrightarrow KR^0(B(TX), S(TX)) \longrightarrow KR^0(B(TX)|_{\partial X}, S(TX)|_{\partial X}) \end{aligned}$$

The symbol σ of an elliptic operator lives in the third group, and Atiyah-Bott construct a lift to the second group from a local boundary condition. (The image of σ in the last group is an obstruction to the existence of local boundary conditions; the image of the first group in the second measures differences of boundary conditions.) The relative boundary conditions on the twisted de Rham complex are *universal*, so the corresponding lift of the symbol occurs in the universal setting. Recall from the proof of Lemma 3.16 the exact sequence (3.15), now extended one step to the left:

$$KR_G^0(i\mathbb{R})_c \longrightarrow KR_{SO_2 \times G}^0(\mathbb{C}^2)_c \longrightarrow KR_{SO_2 \times G}^0(i\mathbb{R}^2)_c \longrightarrow KR_G^0(i\mathbb{R}^2)_c$$

The group G acts trivially in all cases. The Atiyah-Bott procedure applied to the relative boundary conditions gives a lift of $\bar{\sigma}_{\text{univ}} \in KR_{SO_2 \times G}^0(i\mathbb{R}^2)_c$ to $KR_{SO_2 \times G}^0(\mathbb{C}^2)_c$. Recall that $\bar{\sigma}'_{\text{univ}}$, constructed in the proof of Lemma 3.16, is also a lift of $\bar{\sigma}_{\text{univ}}$. But by periodicity we find $KR_G^0(i\mathbb{R})_c \cong KR_G^0(-\mathbb{R})_c \cong KO_G^0(-\mathbb{R})_c \cong KO_G^1(\text{pt})$ which vanishes by [An]. Thus the lift of $\bar{\sigma}_{\text{univ}}$ is unique and $\bar{\sigma}'_{\text{univ}}$ computes the relative twisted de Rham cohomology. This completes the proof.

The Level

A universal orientation induces a level, which is commonly used to identify the theory. One observation arising from this study is that the level is a derived quantity, and it is the universal orientation which determines the theory. We explain the relationship, and deduce the existence of universal orientations, in this subsection.

To begin we recast Definition 3.7 of a universal orientation in a form suited for surfaces with boundary. Consider the diagram

$$(3.20) \quad \begin{array}{ccccc} MTSO_2 \wedge BG & \xrightarrow{q} & (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG & \xrightarrow{r} & MTSO_1 \wedge BG \\ & \searrow \bar{\sigma}_{\text{univ}} & \downarrow \bar{\sigma}'_{\text{univ}} & & \downarrow -\lambda \\ & & ko & \xrightarrow{\tau} & pic_g K \end{array}$$

The top row is a cofibration, the continuation of the top row of (3.15) in the Puppe sequence. Recall that a universal orientation is a null homotopy of $\tau \circ \bar{\sigma}_{\text{univ}} = \tau \circ \bar{\sigma}'_{\text{univ}} \circ q$.

Lemma 3.21. *A universal orientation is equivalent to a map $-\lambda$ in (3.20) and a homotopy from $\tau \circ \bar{\sigma}'_{\text{univ}}$ to $-\lambda \circ r$.*

The proof is immediate. In view of (3.12) and adjunction we can write $\lambda: \Sigma^\infty BG \rightarrow \Sigma \text{pic}_g K$ as a map from the suspension spectrum of BG to the spectrum $\text{pic}_g K$, or equivalently as a map

$$(3.22) \quad \lambda: BG \longrightarrow B \text{Pic}_g K$$

on spaces.

Definition 3.23. *The map (3.22) is the level induced by a universal orientation.*

There is a map $K(\mathbb{Z}, 4) \rightarrow B \text{Pic}_g K$ (see (2.11)) and in some important cases, for example if G is connected and simply connected, the level factors through $K(\mathbb{Z}, 4)$, i.e., the level is a class $\lambda \in H^4(BG)$.

A universal orientation is more than a level: it is a map $-\lambda: MTSO_1 \wedge BG \rightarrow \text{pic}_g K$ together with a homotopy of $-\lambda \circ r$ and $\tau \circ \bar{\sigma}'_{\text{univ}}$ in (3.20). Our next result proves that universal orientations exist.

Theorem 3.24. *There is a canonical universal orientation μ . The corresponding level h is the negative of $BG \rightarrow BO \rightarrow B \text{Pic}_g K$, where the first map is induced from the reduced adjoint representation $\bar{\mathfrak{g}}$ and the second is projection to a Postnikov section.*

Proof. Since complex vector spaces have a canonical K -theory orientation, the composite map $k \longrightarrow ko \xrightarrow{\tau} \text{pic}_g K$ is null, where k is the connective K -theory spectrum. (See the text preceding (2.12).) Therefore, a universal orientation is given by filling in the left dotted arrow in the diagram

$$(3.25) \quad \begin{array}{ccccc} MTSO_2 \wedge BG & \xrightarrow{q} & (MTSO_2, \Sigma^{-1} MTSO_1) \wedge BG & \xrightarrow{r} & MTSO_1 \wedge BG \\ \vdots \downarrow & & \downarrow \bar{\sigma}'_{\text{univ}} & & \downarrow -\lambda \\ k & \xrightarrow{\quad} & ko & \xrightarrow{\tau} & \text{pic}_g K \end{array}$$

and specifying a homotopy which makes the left square commute. There is a natural choice: smash the K -theory Thom class $U: MTU_1 \simeq MTSO_2 \rightarrow k$ with the complexified reduced adjoint representation $\bar{\mathfrak{g}}_{\mathbb{C}}$. This is the universal rewriting of de Rham on a Riemann surface in terms of Dolbeault, at least on the symbolic level. In terms of the proof of Lemma 3.16, the map $\bar{\sigma}_{\text{univ}}$, restricted to $MTSO_2$, is the exterior algebra complex $(\bigwedge^\bullet \mathbb{C}^2, \epsilon)$ in $KR_{SO_2}^0(i\mathbb{R}^2)_c$. Write $\mathbb{R}^2 = L_{\mathbb{R}}$ for the complex line $L = \mathbb{C}$ and $\mathbb{C}^2 \cong \mathbb{R}^2 \otimes \mathbb{C} \cong L \oplus \bar{L}$. Then the symbol complex at $\theta \in i\mathbb{R}^2$,

$$\mathbb{C} \xrightarrow{\epsilon(\theta)} L \oplus \bar{L} \xrightarrow{\epsilon(\theta)} L \otimes \bar{L},$$

is the realification of the complex

$$\mathbb{C} \xrightarrow{\epsilon(\theta)} L$$

which defines the K -theory Thom class. Tensor with the complexified reduced adjoint representation $\bar{\mathfrak{g}}_{\mathbb{C}}$ to complete the argument.

To compute the level of μ we factorize τ as $ko \xrightarrow{\eta} \Sigma^{-1}ko \longrightarrow \text{pic}_g K$, where the first map is multiplication by $\eta: S^0 \rightarrow S^{-1}$ and the second is projection to a Postnikov section. The map η fits into the Bott sequence $k \rightarrow ko \rightarrow \Sigma^{-1}ko$, and so we extend (3.25) to the left:

$$(3.26) \quad \begin{array}{ccccccc} \Sigma^{-1}MTSO_1 \wedge BG & \longrightarrow & MTSO_2 \wedge BG & \xrightarrow{q} & (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG \\ \downarrow \alpha & & \downarrow U \wedge \bar{\mathfrak{g}}_{\mathbb{C}} & & \downarrow \bar{\sigma}'_{\text{univ}} \\ \Sigma^{-1}ko & \longrightarrow & \Sigma^{-2}ko & \longrightarrow & k & \longrightarrow & ko \end{array}$$

The homotopy which expresses the commutativity of the right hand square induces the map α in this diagram, and the map $-\lambda$ induced in (3.25) is the suspension of α . We claim that there is a unique α , up to homotopy, which makes the left square in (3.26) commute. For the difference of any two choices for α is a map $\Sigma^{-1}MTSO_1 \wedge BG \rightarrow \Sigma^{-1}ko$, and the homotopy classes of such maps form the group $KO^1(BG)$ which vanishes [An]. It is easy to find a map α as follows. Since (Lemma 3.10(i)) $\Sigma^{-1}MTSO_1 \simeq S^{-2}$, the upper left map is the inclusion of the bottom cell of $MTSO_2 \wedge BG$. The composite $\Sigma^{-1}MTSO_1 \wedge BG \simeq \Sigma^{-2}BG \rightarrow k$ factors as $\Sigma^{-2}BG \rightarrow \Sigma^{-2}k \rightarrow k$, where the first map is the double desuspension of $\bar{\mathfrak{g}}_{\mathbb{C}}$ and the second Bott periodicity. Choose α to be the double desuspension of $\bar{\mathfrak{g}}$, the *real* reduced adjoint representation. This completes the proof.

Since equivalence classes of universal orientations form a torsor for the group $\mathcal{O}(G)$ in (3.9), the canonical universal orientation identifies the torsor of universal orientations with $\mathcal{O}(G)$. Notice the natural map

$$(3.27) \quad \ell: \mathcal{O}(G) = [\Sigma MTSO_2 \wedge BG, \text{pic}_g K] \longrightarrow [MTSO_1 \wedge BG, \text{pic}_g K] \cong [BG, B \text{Pic}_g K]$$

from universal orientations to levels. If $g \in \mathcal{O}(G)$, then the level of $\mu + g$ is $\ell(g) - h$. If G is connected, simply connected, and simple, then $[BG, B \text{Pic}_g K] \cong H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$ and h is the dual Coxeter number of G times a generator. Then $g \mapsto \ell(g) - h$ is a version of the ubiquitous “adjoint shift”.

Remark 3.28. For any G the top homotopy group of $\text{Map}(\Sigma MTSO_2, \text{pic}_g K)$ and of $B \text{Pic}_g K$ is π_4 , which is infinite cyclic. So there is a homomorphism of $H^4(BG; \mathbb{Z})$ into the domain and codomain of (3.27), and on these subspaces ℓ is an isomorphism. This means that we can *change* a consistent orientation by an element of $H^4(BG; \mathbb{Z})$, and the level changes by the same amount.

The pushforward maps

Suppose we have chosen a universal orientation with level λ . Let X be a compact oriented 2-manifold with boundary. We can work as before with a family of flat connections on X parametrized by a smooth manifold M , but instead for simplicity we work universally on \mathcal{M}_X . As in (3.18) fix a classifying map

$$(3.29) \quad f: \mathcal{M}_X \longrightarrow (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG;$$

then there is an induced classifying map

$$\dot{f} = r \circ f: \mathcal{M}_{\partial X} \longrightarrow MTSO_1 \wedge BG$$

for r the map (3.17). Set $\tau = \dot{f}^*(\lambda)$. Let $t: \mathcal{M}_X \rightarrow \mathcal{M}_{\partial X}$ be the restriction map on flat connections. According to Lemma 3.19 the composition $\tau \circ \bar{\sigma}'_{\text{univ}} \circ f$ is the reduced twisting $\tau_t - (\dim \mathcal{M}_X - t^* \dim \mathcal{M}_{\partial X})$. The homotopy which expresses commutativity of the square in (3.20) gives an isomorphism

$$(3.30) \quad \tau_t - (\dim \mathcal{M}_X - t^* \dim \mathcal{M}_{\partial X}) \xrightarrow{\cong} -t^* \tau.$$

In principle, $\dim \mathcal{M}_X$ and $\dim \mathcal{M}_{\partial X}$ are locally constant functions which vary over the moduli space. However, the Euler characteristic of the deformation complex (3.1) is independent of the connection, and we easily deduce

$$\dim \mathcal{M}_X - t^* \dim \mathcal{M}_{\partial X} \equiv (\dim G) b_0(\partial X) \pmod{2},$$

where b_0 is the number of connected components. (We only track degrees in K -theory modulo two; see (2.5).) According to the discussion in §2 (especially (2.10)), there then is an induced pushforward

$$(3.31) \quad t_*: K^0(\mathcal{M}_X) \longrightarrow K^{\tau + (\dim G) b_0(\partial X)}(\mathcal{M}_{\partial X}).$$

This is the pushforward (1.5) associated to the bordism $X: Y_0 \rightarrow Y_1$ with $Y_0 = \emptyset$ and $Y_1 = \partial X$. The invariant (1.5) is then $t_*(1)$.

Note the special case $\partial X = S^1$. Then $\mathcal{M}_{S^1} = G//G$ is the global quotient stack of the action of G on G by conjugation. The codomain of (3.31) is thus

$$K^{\tau + \dim G}(\mathcal{M}_{S^1}) \cong K^{\tau + \dim G}(G//G) = K_G^{\tau + \dim G}(G).$$

This is the basic space of the two-dimensional TQFT we construct; see §1.

Observe that the universal orientation, in the form described around (3.20), leads to the twist τ in the codomain of (3.31). This is the mechanism which was envisioned in (2.10) when we discussed twistings in general: we have constructed a pushforward from untwisted K -theory to twisted K -theory. The universal orientation neatly accounts for the construction of a Frobenius ring structure on twisted K -theory.

To treat an arbitrary bordism $X: Y_0 \rightarrow Y_1$ we note that the deformation complex at a flat connection a on a principal G -bundle $Q \rightarrow Y_0$ is

$$(3.32) \quad 0 \longrightarrow \Omega_{Y_0}^0(\mathfrak{g}_Q) \xrightarrow{d_a} \Omega_{Y_0}^1(\mathfrak{g}_Q) \longrightarrow 0$$

The operator d_a is skew-adjoint. Therefore, there is a canonical trivialization of the K -theory class of (3.32); e.g., a canonical isomorphism $\ker d_a \cong \operatorname{coker} d_a$. Suppose a classifying map f is given as in (3.29) and let \dot{f}_0, \dot{f}_1 denote its restriction to the boundary connections on Y_0, Y_1 . Set $\tau_i = \dot{f}_i^*(\lambda)$. Then (3.30) and the canonical trivialization of (3.32) on the incoming boundary lead to the desired push-pull map

$$(3.33) \quad K^{\tau_0 + (\dim G) b_0(Y_0)}(\mathcal{M}_{Y_0}) \xrightarrow{s^*} K^{s^*\tau_0 + (\dim G) b_0(Y_0)}(\mathcal{M}_X) \xrightarrow{t_*} K^{\tau_1 + (\dim G) b_0(Y_1)}(\mathcal{M}_{Y_1})$$

This is the map (1.5) with the twistings induced from the universal orientation.

A universal orientation induces consistent orientations on the outgoing restriction maps of bordisms. In other words, if $X: Y_0 \rightarrow Y_1$ and $X': Y_1 \rightarrow Y_2$ are bordisms, then the push-pull maps derived from the diagram

$$\begin{array}{ccccc} & & \mathcal{M}_{X' \circ X} & & \\ & \swarrow r & & \searrow r' & \\ \mathcal{M}_X & & & & \mathcal{M}_{X'} \\ \swarrow s & & \searrow t & \swarrow s' & \searrow t' \\ \mathcal{M}_{Y_0} & & \mathcal{M}_{Y_1} & & \mathcal{M}_{Y_2} \end{array}$$

satisfy

$$(t' r')_* \circ (sr)^* = [t'_* \circ s'^*] \circ [t_* \circ s^*].$$

This follows from Lemma 1.3 and the “Fubini property”

$$(3.34) \quad (t' r')_* = t'_* r'_*$$

of pushforward. The orientation of t induces orientations of r' and $t' r'$, since the diamond is a fiber product. At stake in (3.34) is the consistency of the orientations, which is ensured by the use of a universal orientation. The details of this argument⁴ will be given on another occasion.

One caveat: since $\mathcal{M}_X, \mathcal{M}_{\partial X}$ are stacks we can only pushforward along *representable* maps, and this forces every component of X to have a nonempty outgoing boundary. As mentioned at the end of §1, the partial topological quantum field theory obtained from the push-pull construction extends to a full theory using the invertibility of the (co)pairing attached to the cylinder [FHT3, §17].

4. FAMILIES OF SURFACES, TWISTINGS, AND ANOMALIES

We begin with a general discussion of topological quantum field theories for families. Let F be an n -dimensional TQFT in the most naive sense. Thus F assigns a finite dimensional complex

⁴The main point is that consistent orientations themselves form an *invertible* topological field theory, and these field theories factor through the group completion of bordism, i.e., the Madsen-Tillmann space.

vector space $F(Y)$ to a closed oriented $(n-1)$ -manifold Y and a linear map $F(X): F(Y_0) \rightarrow F(Y_1)$ to a bordism $X: Y_0 \rightarrow Y_1$. In particular, $F(X) \in \mathbb{C}$ if X is closed. Suppose that $\mathcal{Y} \rightarrow S$ is a fiber bundle with fiber a closed oriented $(n-1)$ -manifold. Then the vector spaces assigned to each fiber fit together into a complex vector bundle $F(\mathcal{Y}/S) \rightarrow S$. If $\gamma: [0, 1] \rightarrow S$ is a path, then $\gamma^*\mathcal{Y} \rightarrow [0, 1]$ is a bordism from the fiber $\mathcal{Y}_{\gamma(0)}$ to the fiber $\mathcal{Y}_{\gamma(1)}$. The topological invariance of F shows that $F(\gamma^*\mathcal{Y}): F(\mathcal{Y}_{\gamma(0)}) \rightarrow F(\mathcal{Y}_{\gamma(1)})$ is unchanged by a homotopy of γ , and so $F(\mathcal{Y}/S) \rightarrow S$ carries a natural flat connection. Then a family of bordisms $\mathcal{X} \rightarrow S$ from $\mathcal{Y}_0 \rightarrow S$ to $\mathcal{Y}_1 \rightarrow S$ produces a section $F(\mathcal{X}/S)$ of $\text{Hom}(F(\mathcal{Y}_0/S), F(\mathcal{Y}_1/S)) \rightarrow S$; the topological invariance and gluing law of the TQFT imply that this section is flat. In other words, $F(\mathcal{X}/S) \in H^0(S; \text{Hom}(F(\mathcal{Y}_0/S), F(\mathcal{Y}_1/S)))$. It is natural, then, to postulate that a TQFT in families gives more, namely classes of all degrees:

$$(4.1) \quad F(\mathcal{X}/S) \in H^\bullet(S; \text{Hom}(F(\mathcal{Y}_0/S), F(\mathcal{Y}_1/S))).$$

These classes are required to satisfy gluing laws and topological invariance as well as naturality under base change.

The idea of a TQFT in families—at least in two dimensions—was introduced in the mid 90s. In two dimensions it is often formulated in a holomorphic language (e.g. [KM]), and classes are required to extend to the Deligne-Mumford compactification of the moduli space of Riemann surfaces.

Our push-pull construction works for families of surfaces—with a twist. The purpose of this section is to alert the reader to the twist.⁵

Suppose $\mathcal{X} \rightarrow S$ is a family of bordisms from $\mathcal{Y}_0 \rightarrow S$ to $\mathcal{Y}_1 \rightarrow S$, where $\mathcal{Y}_i \rightarrow S$ are fiber bundles of oriented 1-manifolds. Then the moduli stacks of flat connections form a correspondence diagram over S :

$$\begin{array}{ccc} & \mathcal{M}_{\mathcal{X}/S} & \\ s \swarrow & & \searrow t \\ \mathcal{M}_{\mathcal{Y}_0/S} & & \mathcal{M}_{\mathcal{Y}_1/S} \\ \pi_0 \searrow & & \swarrow \pi_1 \\ & S & \end{array}$$

The push-pull constructs a map from twisted $K(\mathcal{M}_{\mathcal{Y}_0/S})$ to twisted $K(\mathcal{M}_{\mathcal{Y}_1/S})$. We can also work locally over S : the K -theory of the fibers of π_i form bundles of spectra over S and the push-pull gives a map of these spectra. But for our purposes the global push-pull suffices. This construction is a variation of (4.1): we use K -theory rather than cohomology.

The discussion of §3 goes through with one important change. It comes in the paragraph preceding (3.6). For simplicity suppose $\mathcal{Y}_0 = \emptyset$ so that the boundary $\partial\mathcal{X} = \mathcal{Y}_1$ is entirely outgoing. Fix $A \in \mathcal{M}_{\mathcal{X}_s}$ a flat connection on a principal G -bundle $P \rightarrow \mathcal{X}_s$. Then the KO -theory class of the de Rham complex coupled to the reduced adjoint bundle $\bar{\mathfrak{g}}_P = \mathfrak{g}_P - \underline{\dim G}$ computes the difference $T_A \mathcal{M}_{\mathcal{X}_s} - (\dim G) H^\bullet(\mathcal{X}_s)$, where $H^\bullet(\mathcal{X}_s)$ is the real cohomology of \mathcal{X}_s viewed as a class in KO -theory. In §3 we treat $H^\bullet(\mathcal{X}_s)$ as a trivial vector space (there $S = \text{pt}$), but now $H^\bullet(\mathcal{X}_s)$ varies with $s \in S$ and so can give rise to a nontrivial twisting. More precisely, $H^\bullet(\mathcal{X}_s)$ is the fiber at $s \in S$

⁵We thank Veronique Godin for the perspicacious sign question which prompted this exposition.

of a flat vector bundle $\mathcal{H}^\bullet(\mathcal{X}/S) \rightarrow S$. Let $\tau_{\mathcal{X}/S}$ denote the twisting of complex K -theory attached to this bundle. This twisting replaces the degree shift in (3.30) and the pushforward (3.31) is modified to include that extra twist:

$$(4.2) \quad t_*: K^0(\mathcal{M}_{\mathcal{X}/S}) \longrightarrow K^{\tau + (\dim G)} \pi_1^* \tau_{\mathcal{X}/S}(\mathcal{M}_{y_1/S}).$$

The degree shift is now incorporated into the twist $\tau_{\mathcal{X}/S}$, and there may be nontrivial contributions to the twist from w_1 and W_3 of $\mathcal{H}^\bullet(\mathcal{X}/S) \rightarrow S$ as well. (The degree shift and twistings vanish canonically if $\dim G$ is even.)

Example 4.3. Consider the disjoint union X of two 2-disks. The boundary circles are outgoing, as above. Suppose that $\dim G$ is odd. For a single disk the pushforward $t_*(1)$ in (3.31) lands in $K_G^{\tau+1}(G)$ and is the unit $\mathbf{1}$ in the Verlinde ring. Thus for the disjoint union of two disks, $t_*(1)$ is the image of $\mathbf{1} \otimes \mathbf{1}$ under the external product $K_G^{\tau+1}(G) \otimes K_G^{\tau+1}(G) \rightarrow K_{G \times G}^{(\tau, \tau)}(G \times G)$. Now consider the family $\mathcal{X} \rightarrow S$ with fiber X and base $S = S^1$ in which the monodromy exchanges the two disks. The flat bundle $\mathcal{H}^\bullet(\mathcal{X}/S) \rightarrow S$ has rank 2 and nontrivial w_1 . According to (4.2), then, $t_*(1)$ for the family lives in the twisted group $K^{(\tau, \tau) + \pi^* \tau_{\mathcal{X}/S}}(\mathcal{M}_{\partial \mathcal{X}/S})$. On each fiber of $\pi: \mathcal{M}_{\partial \mathcal{X}/S} \rightarrow S$ we recover the class $\mathbf{1} \otimes \mathbf{1}$ above. But upon circling the loop $S = S^1$ this class changes sign in the $\pi^* \tau_{\mathcal{X}/S}$ -twisted K -group. Said differently, the diffeomorphism which exchanges the disks acts by a sign on $\mathbf{1} \otimes \mathbf{1}$. Of course, one might predict this from the sign rule in graded algebra: the Verlinde ring $K_G^{\tau+1}(G)$ is in odd degree, so upon exchanging the factors of $\mathbf{1} \otimes \mathbf{1}$ one picks up a sign. It shows up here as an extra twisting.

This extra twisting is a topological analog of what is usually called an *anomaly* in quantum field theory. In an anomalous theory in n dimensions the partition function on a closed n -manifold, rather than being a complex-valued function on a space of fields, is a section of a complex line bundle over that space of fields. Furthermore, there is a gerbe over the space of fields on a closed $(n-1)$ -manifold, and for a bordism the partition function is a map of the gerbes attached to the boundary. In the homological version described at the beginning of this section, the parameter space S plays the role of the space of fields and for a family of closed n -manifolds the partition function in a non-anomalous theory is an element of $H^\bullet(S; \mathbb{R})$. An anomalous theory would assign a flat complex line bundle $L \rightarrow S$ to the family, and the partition function would live in the twisted cohomology $H^\bullet(S; L) = H^{L+\bullet}(S)$. In the 2-dimensional TQFT we construct using push-pull on K -theory, the extra K -theory twist $\tau_{\mathcal{X}/S}$ is the anomaly; see (4.2). Notice that there is no gerbe attached to a family of 1-manifolds (better: it is canonically trivial). We remark that the anomaly is itself a particular example of an invertible topological quantum field theory.

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